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EFFECTIVE ELASTIC–PLASTIC COMPRESSIBILITY OF POROUS BODIES

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Abstract—In the present paper relations between the hydrostatic pressure and the mean pore volume of a porous body are offered. Especially, attention is drawn to the plastic range of deformation history. Contrary to previous works, the interaction between different pores is attempted to be taken into consideration in a summary manner by using a self-consistent method. For this end the surroundings of a single pore is replaced by a homogeneous material, which is described by the compression-law in question. The results are evaluated numerically and approximate formulae are given. Comparison with experimental results shows partially good coincidence.

1. INTRODUCTION

THE PRESENT work is concerned with the mechanical behaviour of a porous body under hydrostatic pressure p. We want to derive relations between p and the mean volume $\bar{V}_p$ of the pores. The pure material may behave according to linear elasticity, and perfect plasticity, respectively. Here Tresca's yield condition, with yield stress tension σ_0, is applied. For the reason of mathematical simplicity spherical holes are assumed. Further, we restrict ourselves on the quasistatic behaviour. For problems connected with the dynamic compaction see, e.g. [8, 9].

First Torre [1] has offered a solution of the problem under restrictive conditions which can be stated as follows. All the pores have equal size V_p and are encased by the same volume V_M of matter. Further, the same pressure p is acting on the exterior surface of these casings. Finally, elastic deformation is disregarded. The result is

$$\begin{aligned} p &= \tfrac{2}{3}\sigma_0 \ln \frac{V_p + V_M}{V_p} \quad &&\text{for} \quad p \geq \tfrac{2}{3}\sigma_0 \ln \frac{V_p^0 + V_M}{V_p^0} \\ V_p &= V_p^0 &&\text{for} \quad p \leq \tfrac{2}{3}\sigma_0 \ln \frac{V_p^0 + V_M}{V_p^0} \end{aligned} \tag{1}$$

where V_p^0 is the pore volume at vanishing pressure. For the real case (1) may be approximately used replacing V_p, V_p^0, V_M by their mean values $\bar{V}_p, \bar{V}_p^0, \bar{V}_M$. Pompe and Merz [2] have refined this model by assuming a distribution function $f(V_p^0)$. Thus they write

$$\bar{V}_p = \frac{1}{\int_0^\infty f(V_p^0)\,\mathrm{d}V_p^0}\left[\int_0^{V_p(p)} f(V_p^0)V_p^0\,\mathrm{d}V_p^0 + V_p(p)\int_{V_p(p)}^{\infty} f(V_p^0)\,\mathrm{d}V_p^0\right] \tag{1.1}$$

with $V_p(p)$ given by $(1)_1$.

From a physical point of view the most serious lack of these models consists in ruling out the interaction between the pores from the beginning by means of the above mentioned presupposition concerning the pressure. Chu and Hashin [3] have suggested a so called "composite sphere model". In this model the pressure p is acting at infinity and the material is composed of hollow spheres. Every sphere consists of a pore with volume V_p^0 and a casing with volume V_M, where V_M/V_p^0 is constant, everywhere. As has been shown by the authors, their assumptions entail that there is no interaction between the spheres, such that they arrive at the same results as Torre did so (1).

Contrary to the mentioned works, in the present paper interaction between the pores is attempted to be taken into consideration in a summary manner by deriving a constitutive equation

$$\bar{\mathbf{e}} = \varphi(\bar{\mathbf{s}}) \tag{1.2}$$

which relates the mean normal stress $\bar{\mathbf{s}}$ to the mean compressibility $\bar{\mathbf{e}}$. (Remember decompositions like $\sigma_{ij} = s\delta_{ij}+s_{ij}$, $s_{ii} = 0$). The shearing behaviour is assumed to be governed by perfect plasticity being characterized by an effective yield stress σ_0^e and an effective shear modulus G^e which will be fitted to experimental data. (The corresponding constants of the pure material are called σ_0 and G, respectively.) In order to determine (1.2) a self consistent method will be used, similar to that as is known already in connection of the calculation of effective elastic constants. For this end, the inhomogeneous material, surrounding a single pore, is replaced by the so called homogeneous effective material with the effective constants σ_0^e, G^e and the still unknown effective compressibility (1.2):

$$e^e = \varphi(s^e). \tag{1.2a}$$

(The index ^{e} means effective material). The solution of the corresponding elastic–plastic problem will relate the compression of a single pore to the mean hydrostatic stress. Finally, averaging will lead to the determination of the unknown function in (1.2). In detail we shall discuss two models:

(I) The inclusion is simply a pore with initial and final volume V_p^0, V_p, respectively. Beginning with a certain value $p = p^*$ with increasing pressure a plastic zone will grow from the pore surface outward through the effective material.

(II) The inclusion consists of a pore and a casing as described above. Here the plastic zone will run through the casing up to its exterior boundary, since the elastic–plastic behaviour of the effective material is merely described by the assumption of the constitutive relation (1.2). In the framework of this model we do not deal explicitly with plastification of the effective material.

We shall disregard surface effects and, therefore, formally deal with an infinite body; that is, we assume the boundary condition

$$\sigma_{ij}^e(r \to \infty) = \bar{\boldsymbol{\sigma}}_{ij} = \bar{\mathbf{s}}\delta_{ij} = -p\delta_{ij}. \tag{1.3}$$

Because of (1.2) also

$$\varepsilon_{ij}^e(r \to \infty) = \bar{\boldsymbol{\varepsilon}}_{ij} = \bar{\mathbf{e}}\delta_{ij} \tag{1.4}$$

holds. p is the exterior pressure applied on the body. Further, the inclusion can be put into the origin of coordinates such that the whole problem becomes spherically symmetric. Hence, the basic mechanical law reads

$$\frac{d\sigma_r^e}{dr}+\frac{2}{r}(\sigma_r^e-\sigma_t^e) = 0 \tag{1.5}$$

and $v_r^e = v^e(r)$ is the only non-vanishing component of the displacement vector.

In the effective material, so to say, pores and pure matrix are smeared over the body. In this way, we are dealing with the pore volume $V_p^e(r)$ as a continuous function of position, such that any volume element of the effective material at position r may be written as

$$dV_r^e = (V_p^e(r)+V_M^e(r))\,d\gamma. \tag{1.6}$$

Concerning compressibility we disregard the compression of the pure material in comparison with that of the holes. In other words, the pure matrix behaves isochorically, in contrast to the effective material whose compressibility is given by (1.2a). Therefore,

$$e^e(r) = \frac{1}{3}\,\frac{dV_r^e-dV_r^{e0}}{dV_r^{e0}} = \frac{1}{3}\,\frac{V_p^e(r)-\bar{\mathbf{V}}_p^0}{\bar{\mathbf{V}}_p^0+\bar{\mathbf{V}}_M} \tag{1.7}$$

$$e^e(r) = \frac{\bar{\alpha}}{3}\left(\frac{V_p^e(r)}{\bar{\mathbf{V}}_p^0}-1\right)$$

and

$$\bar{\mathbf{e}} = \frac{\bar{\alpha}}{3}\left(\frac{\bar{\mathbf{V}}_p}{\bar{\mathbf{V}}_p^0}-1\right)$$

are valid, where

$$\bar{\alpha} = \frac{\bar{\mathbf{V}}_p^0}{\bar{\mathbf{V}}_p^0+\bar{\mathbf{V}}_M} \tag{1.8}$$

is a mean porosity. From (1.2a) with regard to (1.7) one gets

$$\frac{V_p^e(r)}{\bar{\mathbf{V}}_p^0} = 1+\frac{3}{\bar{\alpha}}\varphi(s^e).$$

For more convenience it is preferable to replace the right-hand side of this equation by a function f in the following manner:

$$\frac{V_p^e(r)}{\bar{\mathbf{V}}_p^0} = f\left(\frac{3}{2}\,\frac{(-s^e(r))}{\sigma_0^e}\right) \qquad \frac{df}{d(-s^e)} \leq 0. \tag{1.9}$$

By means of the abbreviations

$$y = \frac{\bar{\mathbf{V}}_p}{\bar{\mathbf{V}}_p^0}, \qquad z = \frac{3p}{2\sigma_0^e} \tag{1.10}$$

(1.9) turns into

$$y = f(z) \tag{1.11}$$

for large distances from the inclusion ($r \to \infty$).

Thus the determination of the unknown function f solves the initial problem. Now, the solution of the above mentioned elastic–plastic problem looks generally like

$$V_p = V_p(V_p^0, V_M, \bar{\mathbf{V}}_p^0, \bar{\mathbf{V}}_M, \bar{\mathbf{V}}_p, p_1[f]) \tag{1.12}$$

where $[f]$ indicates a possibly occurring functional dependency. After averaging over V_p^0, V_M and on account of (1.10) there yields a functional equation for the unknown f

$$\frac{\bar{\mathbf{V}}_p}{\bar{\mathbf{V}}_p^0} = y = f(z) = F\left(\frac{\bar{\mathbf{V}}_p}{\bar{\mathbf{V}}_p^0}, \bar{\alpha}, z, [f]\right) \tag{1.13}$$

(F stems from averaging the right-hand side of (1.12).)

In the general case this functional equation is too complicated for being discussed without further approximations. Throughout this paper

$$|e^e(r)| = \frac{\bar{\alpha}}{3}\left(1 - \frac{V_p^e(r)}{\bar{\mathbf{V}}_p^0}\right) \ll 1 \tag{1.14}$$

will be proposed. This requirement does not prevent the pores to vanish ($\bar{\mathbf{V}}_p = 0$), so far as $\bar{\alpha}$ has been chosen sufficiently small.

Since $V_p^e(r)$ will turn out to decrease with increasing r it is enough demanding

$$|\bar{e}| = \frac{\bar{\alpha}}{3}(1-y) \ll 1. \tag{1.15}$$

In particular, (1.14) leads to the replacement of (1.7) by

$$\begin{aligned} 3e^e(r) &= \bar{\alpha}\left(f\left(-\frac{3}{2}\frac{s^e(r)}{\sigma_0^e}\right) - 1\right) \\ &\approx \frac{\mathrm{d}V_r^e - \mathrm{d}V_r^{e0}}{\mathrm{d}V_r^e} \end{aligned} \tag{1.16}$$

where, moreover, (1.9) has been substituted.

2. MODEL I. PURLY ELASTIC DEFORMATION

We employ linear elasticity and notice

$$s_{ij}^e = 2G^e e_{ij}^e \tag{2.1}$$

$$e_r^e = \frac{2}{3}\left(\frac{\mathrm{d}v^e}{\mathrm{d}r} - \frac{v^e}{r}\right) \qquad e_t^e = -\tfrac{1}{2}e_r^e \tag{2.2}$$

$$e^e = \frac{1}{3}\left(\frac{\mathrm{d}v^e}{\mathrm{d}r} + 2\frac{v^e}{r}\right). \tag{2.3}$$

Thus (1.5) goes over to

$$\left(4G^e+\frac{ds^e}{de^e}\right)\left(\frac{d^2v^e}{dr^2}+\frac{2}{r}\frac{dv^e}{dr}-2\frac{v^e}{r^2}\right)=0. \tag{2.4}$$

On account of (1.16), $(1.9)_2$ the first factor cannot vanish, and hence, we obtain

$$v^e(r)=C_1r+C_2r^{-2} \tag{2.5}$$

from which it follows

$$s_r^e(r)=-4G^eC_2r^{-3}=-2s_t^e(r) \tag{2.6}$$

$$e^e=C_1=\bar{\mathrm{e}}\qquad s^e=-p \tag{2.7}$$

where (1.3), (1.4) and (1.9) have been taken into account. On the surface of the hole with radius r_i the condition

$$\sigma_r^e=s_r^e+s^e=0\quad\text{for } r=r_i \tag{2.8}$$

must be required which furnishes

$$C_2=-\frac{p}{4G^e}r_i^3.$$

Thus we arrive at

$$v^e(r)=r\left[\bar{\mathrm{e}}-\frac{p}{4G^e}\left(\frac{r_i}{r}\right)^3\right]. \tag{2.9}$$

In the framework of linear elasticity r_i can be taken as the initial radius of the hole, that of the deformed one then being $r_i+v^e(r_i)$. Therefore and because of (2.9), (1.7) an equation of type (1.12) is easily found, namely

$$\begin{aligned}\frac{V_p}{V_p^0}&=\frac{(r_i+v^e(r_i))^3}{r_i^3}\approx 1+3\frac{v^e(r_i)}{r_i}\\&=1-\bar{\boldsymbol{\alpha}}(1-y)-\frac{3p}{4G^e}.\end{aligned} \tag{2.10}$$

In this case the average is trivially performed. With the notation

$$k=\frac{\sigma_0^e}{2G^e} \tag{2.11}$$

the final result reads

$$y=f(z)=1-\frac{k}{1-\bar{\boldsymbol{\alpha}}}z. \tag{2.12}$$

This is our desired material relation in the case that the exterior pressure p is too small for plastification. Let us show that plastification starts at the surface of the hole when the pressure p is

$$p^*=\tfrac{2}{3}\sigma_0^e. \tag{2.13}$$

It is enough to observe

$$\sigma_r^e - \sigma_t^e = \tfrac{3}{2} s_r^e = \tfrac{3}{2}(r_i/r)^3 p \tag{2.14}$$

such that the maximal shear stress occurs, in fact, at $r = r_i$. Equating (2.14) with σ_0^e, according to Tresca's yield condition, for $r = r_i$ proves (2.13). Therefore, (2.12) is valid for $p \leq p^*$ i.e. $z \leq z^* = 1$. $z = z^*$ corresponds to

$$y = y^* = 1 - \frac{k}{1-\bar{\alpha}}. \tag{2.15}$$

Finally, we remark that for most of the real materials k can be estimated as $k < 0{\cdot}01$, therefore

$$k \ll 1 \tag{2.16}$$

holds.

3. MODEL I. ELASTIC–PLASTIC DEFORMATION

If $p > p^*$ there exists a plastic–elastic boundary, say, at $r = \rho$. Hence, the material deforms elastically within the region $r \geq \rho$. This means, in turn, that (2.1) to (2.7) remain valid, but, instead of (2.8), we must take into account the boundary condition

$$\sigma_0^e = \tfrac{3}{2} s_r^e(\rho) = -6G^e C_2 \rho^{-3} \tag{3.1}$$

Consequently, (see 2.5, 2.6 and 2.7)

$$\begin{aligned} v^e(r) &= r\left[\bar{\mathrm{e}} - \frac{\sigma_0^e}{6G^e}\left(\frac{\rho}{r}\right)^3\right] \\ \sigma_r^e(r) &= -p + \tfrac{2}{3}\sigma_0^e\left(\frac{\rho}{r}\right)^3. \end{aligned} \tag{3.2}$$

In the plastic region $r_i \leq r \leq \rho$ Tresca's yield condition

$$\sigma_r^e - \sigma_t^e = \sigma_0^e \tag{3.3}$$

is inserted into the basic law (1.5) which immediately integrates as

$$\sigma_r^e(r) = -\tfrac{2}{3}\sigma_0^e \ln(r/r_i)^3 \tag{3.4}$$

$$s^e(r) = \tfrac{1}{3}(\sigma_r^e + 2\sigma_t^e) = -\tfrac{2}{3}\sigma_0^e(1 + \ln(r/r_i)^3). \tag{3.5}$$

Here, in contrast to the previous section, $V_p = \frac{4}{3}\pi r_i^3$ holds, i.e. the condition (2.8) is fulfilled on the variable boundary $r = r_i$. Because the plastic problem is statically determined there is no need for requiring the deformation to be small within the whole plastic region but only in the vicinity of $r = \rho$. In this way let us notice

$$\begin{aligned} \frac{v^e(\rho)}{\rho} &= \frac{\rho - \rho^0}{\rho} = 1 - \frac{\rho^0}{\rho} = 1 - \sqrt[3]{\left(\frac{V_\rho^0 + V_p^0}{V_\rho + V_p}\right)} \\ V_\rho^0 &= \frac{4\pi}{3}(\rho^{03} - r_i^{03}) \qquad V_\rho = \frac{4\pi}{3}(\rho^3 - r_i^3) \end{aligned} \tag{3.6}$$

where ρ^0 means the initial position of those mass points which were situated at $r = \rho$ when the pressure p is acting. Then, because of (3.2, 3.6 and 1.15)$_1$, the condition that v^e be continuous at $r = \rho$ furnishes

$$\frac{V_\rho^0 + V_p^0}{V_\rho + V_p} = \left(1 + \frac{\sigma_0^e}{6G^e} + \tfrac{1}{3}\bar{\alpha}(1-y)\right)^3 \equiv w(y). \tag{3.7}$$

The second boundary condition, which concerns the continuity of σ_r^e at $r = \rho$, gives

$$\frac{3p}{2\sigma_0^e} = z = 1 + \ln \frac{V_\rho + V_p}{V_p}. \tag{3.8}$$

In order to eliminate the unknown quantities V_ρ, V_ρ^0 equation (1.16) is integrated over the volume V_ρ. With the help of the substitution (see 3.5)

$$t = -\frac{3s^e(r)}{2\sigma_0^e} = 1 + \ln \frac{r^3}{r_i^3}$$

and observing (3.8) the mentioned integration leads to

$$V_\rho - V_\rho^0 = -\bar{\alpha} V_p \left(e^{z-1} - 1 - \int_1^z e^{t-1} f(t)\,\mathrm{d}t\right). \tag{3.9}$$

Combining this relation with (3.7) renders the desired elimination possible and establishes again an equation of type (1.12). Instead of stating such an equation, let us immediately proceed to the averaged one of type (1.13) since the last step is again trivial in the present case.

$$y\left[(w(y) - (1+\bar{\alpha}))\, e^{z-1} + (1+\bar{\alpha}) + \bar{\alpha} \int_1^z e^{t-1} f(t)\,\mathrm{d}t\right] = 1. \tag{3.10}$$

Because of $y = f(z)$ (1.11), in fact, we are faced with an integral equation for $f(z)$ whose determination is our actual aim. It cannot be achieved deriving explicitly $f(z)$ itself. However, its inverse function can be found. Denoting

$$x = e^{z-1} = e^{f^{-1}(y)-1} = x(y) \tag{3.11}$$

we get

$$x(y) = \frac{1}{g(y)}\left[g(y_1)\exp\left(-\bar{\alpha}\int_y^{y_1} \frac{\mathrm{d}s}{g(s)}\right) + \int_y^{y_1} \frac{1}{t^2} \exp\left(-\bar{\alpha}\int_y^t \frac{\mathrm{d}s}{g(s)}\right)\mathrm{d}t\right] \tag{3.12}$$

with the abbreviation

$$g(y) = w(y) - (1+\bar{\alpha}) + \bar{\alpha} y \tag{3.13}$$

and y_1 being determined by

$$w(y_1)\,.\,y_1 = 1. \tag{3.14}$$

Obviously, $y = y_1$ corresponds to $z = z^* = 1$. In order to obtain (3.12) to (3.14) most conveniently realize that by differentiation of (3.10) with respect to x there occurs a linear differential equation for $x(y)$.

Observing the inequalities (1.15 and 2.16) it is seen that in first order

$$g(y) = k \tag{3.15}$$

holds. With the same accuracy y_1 can be replaced by $y^* = 1-k/(1-\bar{\alpha})$, which corresponds to $z = 1$ in the elastic case treated in Section 2. In the frame of that approximation (3.12) is simplified as

$$x(y) = \frac{1}{ky}+\exp\left[-\frac{\bar{\alpha}}{k}(y^*-y)\right]\left(1-\frac{1}{ky^*}\right)+\frac{\bar{\alpha}}{k^2}\exp\left(\frac{\bar{\alpha}}{k}y\right)\left[Ei\left(-\frac{\bar{\alpha}}{k}y\right)-Ei\left(-\frac{\bar{\alpha}}{k}y^*\right)\right] \quad (3.16)$$

$$Ei(-t) = \int_{-\infty}^{-t}\frac{e^s}{s}\,ds.$$

Further simplification is achieved by use of an asymptotic formula for the logarithmic integral Ei (see [4]), which is justified for $y \gtrsim 10(k/\bar{\alpha})$

$$x = \frac{1}{\bar{\alpha}y^2}\left(1-\frac{2}{\bar{\alpha}y}k\right)-\exp\left[-\frac{\bar{\alpha}}{k}(y^*-y)\right]\left(\frac{1}{\bar{\alpha}y^{*2}}\left(1-\frac{2k}{\bar{\alpha}y^*}\right)-1\right). \quad (3.17)$$

Here the second term can be disregarded because of (2.16), unless $y \approx y^*$. Hence, let us assert

$$x(y) = \frac{1}{\bar{\alpha}y^2}\left(1-\frac{2k}{\bar{\alpha}y}\right) \quad \text{for } y^*-\frac{10}{\bar{\alpha}}k \gtrsim y \gtrsim \frac{10}{\bar{\alpha}}k \quad (3.18)$$

or even

$$x(y) = \frac{1}{\bar{\alpha}y^2} \quad \text{for } y^*-\frac{10}{\bar{\alpha}}k \gtrsim y \gtrsim \frac{20}{\bar{\alpha}}k. \quad (3.19)$$

In the other case, treating both $|y-y^*|$ and k, as small quantities, remodels (3.17) into

$$\left.\begin{aligned} x &= A+\frac{2}{\bar{\alpha}}(y^*-y)-(A-1)\exp\left[-\frac{\bar{\alpha}}{k}(y^*-y)\right] \quad \text{for } y^* \gtrsim y \gtrsim y^*-0{\cdot}1 \\ A &= \frac{1}{\bar{\alpha}}\left(1-2k\frac{1-2\bar{\alpha}}{\bar{\alpha}(1-\bar{\alpha})}\right) \equiv \frac{1}{\bar{\alpha}y^{*2}}[1-2k/(\bar{\alpha}y^*)]. \end{aligned}\right\} \quad (3.20)$$

4. MODEL I. UNLOADING

In order to prepare the discussion of unloading, first of all tension–loading may be briefly considered. For this end it is enough to repeat the previous calculations with the following substitutions.

$$\left.\begin{aligned} &p \to -p, \qquad \sigma_0^e \to -\sigma_0^e \\ &z \to z, \qquad x \to x, \qquad k \to -k \end{aligned}\right\} \quad (4.1)$$

The resulting relation is sketched in Fig. 1. Differentiating (3.10) with respect to x and using (3.15) leads to the maximum of tension $z(y)$

$$x_c y_c^2 \bar{\alpha} = 1, \qquad x_c = e^{z_c-1}. \quad (4.2)$$

Here x_c and y_c may be determined by means of (3.20), (4.1), because it is to be expected that $|y_c-y^*| \ll 1$ holds. The result is:

$$x_c = \frac{1}{\bar{\alpha}}(1-2Bk), \qquad y_c = 1+Bk, \qquad B = \frac{1}{1-\bar{\alpha}}+\frac{1}{\bar{\alpha}}\ln\left(\frac{\bar{\alpha}(1-\bar{\alpha})}{2k}\right). \quad (4.3)$$

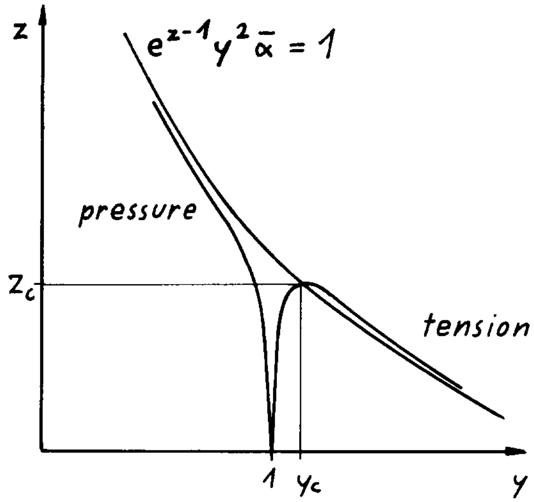

FIG. 1. Stress–strain relation according to model I.

An estimate of B gives

$$0 < B < 20 \tag{4.4}$$

for $k = 0{\cdot}005$ and $0{\cdot}1 \leq \bar{\alpha} \leq 0{\cdot}9$ and therefore $|y_c - y^*| < 0{\cdot}1 \ll 1$ holds, indeed. Now let us turn to the investigation of unloading. When after application of the external pressure p this load is removed again, the question arises to what extent the mean pore volume will form back. For an answer we submit the state of compression (with boundary condition $\sigma_r^e(\infty) = -p$) to an additional tensile stress with $\sigma_r'^e(\infty) = p$. For the sum $\sigma_{ij}''^e = \sigma_{ij}^e + \sigma_{ij}'^e$ the following equations hold:

$$\begin{aligned} 0 = \sigma''^e_{ij,j} &= \sigma^e_{ij,j} + \sigma'^e_{ij,j} = \sigma'^e_{ij,j} \\ \sigma_r''^e - \sigma_t''^e &= -\sigma_0^e \quad \text{within the plastic zone).} \\ \sigma_r''^e(\infty) &= 0. \end{aligned} \tag{4.5}$$

By the help of $\sigma_r^e - \sigma_t^e = \sigma_0^e$ and $(4.5)_2$ it follows

$$\sigma_r'^e - \sigma_t'^e = -2\sigma_0^e \quad \text{(within the plastic zone).} \tag{4.6}$$

It is seen, that σ'_{ij} is governed by the same relations as in the case of tension–loading, except of (4.6), where σ_0^e is replaced by $2\sigma_0^e$ (cf. Prager, Hodge [5]). Hence the results at the beginning of this section may be applied with the following re-interpretation of the occurring quantities, however.

$$\begin{gathered} \sigma_0^e \to 2\sigma_0^e \\ z \to z/2 \qquad x \to x' = e^{z/2-1} \qquad k \to 2k. \end{gathered} \tag{4.7}$$

Moreover, the new initial volume is just the final volume of loading, i.e. one has to replace V_p^0, $\bar{\mathbf{V}}_p^0$ by V_p, $\bar{\mathbf{V}}_p$ respectively, while instead of V_p, $\bar{\mathbf{V}}_p$ there appear the final volumes of unloading, say V'_p, $\bar{\mathbf{V}}'_p$. Consequently the following substitutions are required:

$$\bar{\alpha} = \frac{\bar{\mathbf{V}}^0_p}{\bar{\mathbf{V}}^0_p + \bar{\mathbf{V}}_M} \to \bar{\alpha}' = \frac{\bar{\mathbf{V}}_p}{\bar{\mathbf{V}}_p + \bar{\mathbf{V}}_M} = \frac{y}{y + (1-\bar{\alpha})/\bar{\alpha}} \tag{4.8}$$
$$y = \bar{\mathbf{V}}_p/\bar{\mathbf{V}}^0_p \to y' = \bar{\mathbf{V}}'_p/\bar{\mathbf{V}}_p$$

$y' = y'(y, \bar{\alpha}, z)$ is the quantity to be determined. Since a general computation of y' is too complicated we restrict ourselves to an estimate of y' in comparison to y. For purely elastic unloading from Section 2 and by the help of (4.1, 4.7 and 4.8) it follows:

$$0 \le z \le 2$$
$$y' = 1 + \frac{k}{1-\bar{\alpha}'}z \qquad 1 \le y' \le y^{*\prime} = 1 + \frac{2k}{1-\bar{\alpha}'}. \tag{4.9}$$

In the case of elastic loading, i.e. $0 \le z \le 1$ we can substitute (2.12) into (4.8) to obtain

$$y \, . \, y' = 1 + 0(k^2)$$

which shows in the frame of our approximations that for elastic compression the original form is fully recovered by unloading as it must be so.

Let us turn to the case of elastic–plastic loading but purely elastic unloading i.e. $1 \le z \le 2$. Since by means of (4.8) $0 < \bar{\alpha}' < \bar{\alpha} < 1$ we conclude

$$y' < 1 + \frac{1}{1-\bar{\alpha}}kz < 1 + \frac{2k}{1-\bar{\alpha}} \tag{4.10}$$

from which it is seen that for $z = 2$, already, recovery is negligible in comparison with the plastic compression.

In the case of elastic–plastic unloading $(2 \le z)$ it is to be expected that $x' < x'_c$ holds. $x' < x'_c$ is tantamount to $y' \le y'_c$ and because of (4.3) and (4.4) recovery again turns out to be negligible. Concerning the validity of $x' < x'_c = x'_c[\bar{\alpha}'(y)]$, the following arguments may be offered:

On account of (4.4) x'_c may be replaced by $1/\bar{\alpha}'$, and computing this quantity according to (4.8), we restrict ourselves to sufficiently large plastic deformations such that (3.19) can be used. Under these circumstances (and observing $x = e^{z-1}$, $x' = e^{z/2-1}$) the validity of $x' < x'_c$ turns out to be equivalent with that of

$$e^{z/2}h(\bar{\alpha}) < e \tag{4.11}$$

where $h(\bar{\alpha}) = 1 - (1-\bar{\alpha})\sqrt{(e/\bar{\alpha})}$ is a monotonically increasing function, whose zero is $\bar{\alpha}_0 \approx 0{\cdot}55$.

Therefore (4.11) is valid for arbitrary z so far as $\bar{\alpha} \le \bar{\alpha}_0$, whilst in the case of $\bar{\alpha}_0 < \bar{\alpha} < 1$ the inequality necessitates $z < 2 - 2\ln[h(\bar{\alpha})]$. Such a restriction has no physical meaning. However, we need not bother with it. Namely, the contrary case can be ruled out because it violates our fundamental assumption (1.15):

$$e^{z/2} = \sqrt{(e \, . \, x)} = \sqrt{\left(\frac{e}{\bar{\alpha}}\right)\frac{1}{y}} > \frac{e}{h(\bar{\alpha})}$$

furnishes

$$\tfrac{1}{3}\bar{\alpha}(1-y) \ge \tfrac{1}{3}[1 - \sqrt{(\bar{\alpha}/e)}] > \tfrac{1}{3}(1 - 1/\sqrt{e}) > 0{\cdot}13$$

which cannot claim to be much smaller than unity, in contrast to (1.15).

5. MODEL II

The new model concerns the effective material with an inclusion consisting of a pore (as above) but encased by pure incompressible material with volume

$$V_M = \frac{4\pi}{3}(r_a^3 - r_i^3). \tag{5.1}$$

Deformation, stress and material constants of that casing are denoted by the same signs as above but, in contrast to the effective material, without the index ^{e}.

When the pure material is deformed according to linear elasticity, we may start from the condition of incompressibility

$$e = \frac{\mathrm{d}v}{\mathrm{d}r} + 2\frac{v}{r} = 0$$

such that

$$\begin{aligned} &v = Cr^{-2} \\ &\sigma_r = s - 4GCr^{-3} \qquad \sigma_t = s + 2GCr^{-3} \quad r_i \le r \le r_a \\ &s(r) = s = \text{const.} \end{aligned} \tag{5.2}$$

where the last assertion follows from the basic mechanical law. (1.5 without the index ^{e}). In the effective region $r_a \le r$, obviously, the solution (2.5, 2.6, 2.7), is valid. The constants of integration C_2, C, s are determined by means of the boundary conditions:

$$\begin{aligned} \sigma_r(r_i) &= 0 \\ \sigma_r(r_a) &= \sigma_r^e(r_a) \\ v(r_a) &= v^e(r_a). \end{aligned} \tag{5.3}$$

The result reads

$$\begin{aligned} C_2 &= -[4r_a^{-3}(G^e - G) + 4Gr_i^3]^{-1}\left[p + 4G\bar{e}\left(\frac{r_a^3}{r_i^3} - 1\right)\right] \\ C &= -[4r_a^{-3}(G^e - G) + 4Gr_i^3]^{-1}[p - 4\bar{e}G^e] \\ s &= -p + 4G\bar{e} + 4C_2 r_a^{-3}(G - G^e). \end{aligned} \tag{5.4}$$

A relation of type (1.12) is found as

$$\frac{V_p}{V_p^0} = 1 + 3\frac{v(r_i)}{r_i} = 1 - [G/G^e - \alpha(G/G^e - 1)]^{-1}\left[\frac{3p}{4G^e} + \bar{\alpha}(1 - y)\right]. \tag{5.5}$$

Here in analogy to (1.8) the definition

$$\alpha = \frac{V_p^0}{V_p^0 + V_M} \tag{5.6}$$

has been introduced.

In a similar way as at the end of Section 2 it is seen that plastification begins at $r = r_i$ when p takes the value

$$p^* = \frac{4}{3}G^e\left[\frac{\sigma_0}{2G}\left(\frac{G}{G^e}-\alpha\left(\frac{G}{G^e}-1\right)\right)-\bar{\alpha}(1-y)\right] \tag{5.7}$$

which corresponds to

$$\frac{V_p^*}{V_p^0} = 1-\frac{\sigma_0}{2G}. \tag{5.8}$$

Now for $p > p^*$ there occurs an elastic–plastic boundary at $r = \rho < r_a$. In this case, for $r \geq r_a$ the situation remains unaltered, solution (5.2) holds in the region $\rho \leq r \leq r_a$ while

$$\sigma_r = -2\sigma_0 \ln\frac{r}{r_i}$$
$$\sigma_r-\sigma_t = \sigma_0 \quad \text{holds for } r_i \leq r \leq \rho. \tag{5.9}$$

The requirements $(\sigma_r-\sigma_t)_{r=\rho+0} = \sigma_0$, $v(r_a^0) = v^e(r_a^0)$, $\sigma_r(r_a^0) = \sigma_r^e(r_a^0)$* determine the constants of integration

$$C = -\frac{\sigma_0}{6G}\rho^3$$
$$C_2 = -\bar{\varepsilon}(r_a^0)^3-\frac{\sigma_0}{6G}\rho^3 \tag{5.10}$$
$$s = -p+4G^e\bar{\varepsilon}+\frac{2}{3}\frac{\rho^3}{(r_a^0)^3}\sigma_0\left(\frac{G^e}{G}-1\right)$$

In order to eliminate ρ we use (analogously to (3.6))

$$\left(1-\frac{v}{r}\right)^3_{r=\rho-0} = \frac{V_\rho+V_p^0}{V_\rho+V_p} = \left(1-\frac{v}{r}\right)^3_{r=\rho+0} = \left(1+\frac{\sigma_0}{6G}\right)^3 \approx 1+\frac{\sigma_0}{2G}$$

which furnishes

$$V_\rho = \frac{2G}{\sigma_0}\left[V_p^0-V_p\left(1+\frac{\sigma_0}{2G}\right)\right] \tag{5.11}$$

$$V_p+V_\rho = \frac{4\pi}{3}\rho^3 = \frac{2G}{\sigma_0}(V_p^0-V_p). \tag{5.12}$$

With the aid of all these relations, the requirement that σ_r be continuous at $r = \rho$ leads to an implicit equation of type (1.12), namely

$$\frac{3p}{2\sigma_0} = 1+\ln\left(\frac{2G}{\sigma_0}\frac{V_p^0-V_p}{V_p}\right)+\frac{2G^e}{\sigma_0}\left[\alpha\left(1-\frac{V_p}{V_p^0}\right)-\bar{\alpha}\left(1-\frac{\bar{V}_p}{\bar{V}_p^0}\right)\right]-\frac{2G}{\sigma_0}\alpha\left(1-\frac{V_p}{V_p^0}\right) \tag{5.13}$$

where squares and higher powers of $\sigma_0/(2G)$ have been disregarded. (cf. 2.16).

Substitution of (5.7 and 5.8) into (5.13) shows that both, the purely elastic and the elastic–plastic, solutions fit well together.

* The meaning of r_a^0 is quite analogous to r_i^0.

On the other hand $V_p = V_M$ means total plastification of the casing. According to (5.12 and 5.13) it happens for the pore volume and pressure, respectively, given by

$$\frac{V_p^{**}}{V_p^0} = 1-\frac{\sigma_0}{2G}\frac{1}{\alpha} \qquad \frac{3p^{**}}{2\sigma_0} = \ln\left(\frac{1}{\alpha-\sigma_0/(2G)}\right)+\frac{2G^e}{\sigma_0}\left[\frac{\sigma_0}{2G}-\bar{\alpha}\left(1-\frac{\bar{V}_p}{\bar{V}_p^0}\right)\right]. \tag{5.14}$$

Finally, when $p > p^{**}$, we do not deal explicitly with plastification of the effective material in the framework of the discussed model.

Hence, again the same solution as above is taken for the effective region, while (5.9) is valid for $r_i \le r \le r_a$. The only unknown constant of integration, C_2, is determined by the condition $v^e(r_a) = v(r_a)$, and from the continuity of σ_r at the boundary there yields the implicite equation of type (1.12):

$$z_1 \equiv \frac{3p}{2\sigma_0} = \frac{1}{k_1}\left[\alpha\left(1-\frac{V_p}{V_p^0}\right)-\bar{\alpha}\left(1-\frac{\bar{V}_p}{\bar{V}_p^0}\right)\right]+\ln\left(1+\frac{1-\alpha}{\alpha}\frac{V_p^0}{V_p}\right). \tag{5.15}$$

Here $k_1 = \sigma_0/(2G^e) \ll 1$ has been used, (which together with (1.15) makes $\alpha/3(1-V_p/V_p^0)$ a small quantity). Fitting to the previous solution at $p = p^{**}$ is ensured. Summarizing let us assert:

Solution (5.5) is valid for $0 \le p \le p^*$.
Solution (5.13) is valid for $p^* \le p \le p^{**}$.
Solution (5.15) is valid for $p^{**} \le p$.

6. MODEL II. APPROXIMATIVE EVALUATION

The serious disadvantage of the discussed model in comparison to the first one consists of the impossibility of straightforward proceeding from (5.5, 5.13 and 5.15), respectively, to the desired average equations of type (1.13).

A considerable simplification is achieved in the special case

$$\alpha = \bar{\alpha} \tag{6.1}$$

which is equivalent to

$$V_M = V_p^0\left(\frac{1}{\bar{\alpha}}-1\right). \tag{6.2}$$

Then (5.5) and (5.7) go over to

$$y = 1-\frac{1}{1-\bar{\alpha}}\frac{3p}{4G} \tag{6.3}$$

$$p^* = \tfrac{2}{3}\sigma_0(1-\bar{\alpha}). \tag{6.4}$$

Further, under the supposition (6.1) the relations (5.13 and 5.15) take the form

$$\frac{3p}{2\sigma_0} = \psi\left(\frac{V_p}{V_p^0}\right), \quad \text{i.e.} \quad V_p = V_p^0\psi^{-1}\left(\frac{3p}{2\sigma_0}\right)$$

such that performing the average becomes again trivial:

$$\frac{3p}{2\sigma_0} = \psi(y)$$

or explicitly:

$$\frac{3p}{2\sigma_0} = 1+\ln\left(\frac{2G}{\sigma_0}\frac{1-y}{y}\right)-\frac{2G}{\sigma_0}\bar{\alpha}(1-y) \qquad p^* \le p \le p^{**} \tag{6.5}$$

$$\frac{3p}{2\sigma_0} = \ln\left(1-\frac{1-\bar{\alpha}}{\bar{\alpha}}\frac{1}{y}\right) = \ln\left(\frac{\bar{V}_p+\bar{V}_M}{\bar{V}_p}\right) \qquad p^{**} \le p \tag{6.6}$$

respectively. The pressure p^{**} goes over to

$$\frac{3p^{**}}{2\sigma_0} = \ln\left(\frac{1}{\bar{\alpha}}\left(1+\frac{\sigma_0}{2G}\frac{1}{\bar{\alpha}}\right)\right).$$

(6.6) is Torre's formula (which has been recalculated by Chu and Hashin [3]). As already pointed out on that occasion, the background of it is disregarding any interaction between the pores. It can be argued that the same objection to (6.4) and (6.5) is justified.

Now let us look for another possibility of an approximative evaluation. A suitable approximation seems to consist of disregarding the elastic deformation of the casing, i.e. treating it as a rigid plastic body. Formally, such a treatment amounts to $(\sigma_0/2G) \to 0$, which limiting process transforms (5.8 and (5.14) into

$$\frac{V_p^*}{V_p^0} = \frac{V_p^{**}}{V_p^0} = 1 \qquad \frac{3p^{**}}{2\sigma_0} = -\frac{1}{k_1}\bar{\alpha}\left(1-\frac{\bar{V}_p}{\bar{V}_p^0}\right)+\ln\frac{1}{\alpha} \tag{6.7}$$

The assertion at the end of Section 5 is now modified as

$$\frac{V_p}{V_p^0} = 1 \qquad 0 \le p \le p^{**} \tag{6.8}$$

$$z_1 = \frac{3p}{2\sigma_0} = \frac{1}{k_1}\left[\alpha\left(1-\frac{V_p}{V_p^0}\right)-\bar{\alpha}\left(1-\frac{\bar{V}_p}{\bar{V}_p^0}\right)\right]+\ln\left(\frac{V_p+V_M}{V_p}\right) \qquad p^{**} \le p. \tag{6.9}$$

However, p^{**} depends on α and takes thus different values for different inclusions, while, on the other hand, the external pressure is a given parameter. Therefore, the question, whether or not any pore is deformed, should be decided by means of a remodeled criterion. For this end consider

$$\alpha^{**} = \exp\left[-\left(\frac{3p}{2\sigma_0}+\frac{1}{k_1}\bar{\alpha}\left(1-\frac{\bar{V}_p}{\bar{V}_p^0}\right)\right)\right]. \tag{6.10}$$

As is immediately seen, inclusions with $\alpha = V_p^0/(V_M+V_p^0) \le \alpha^{**}$ behave rigidly, inclusions with $\alpha > \alpha^{**}$ deform plastically according to (6.9). In the V_M, V_p^0-plane these ranges may be denoted by B_1 and B_2, respectively.

Let us call the inverse function of (6.9)

$$V_p = \Phi(V_p^0, V_M, \bar{V}_p, \bar{\alpha}, p) \tag{6.11}$$

and be $w(V_p^0, V_M)$ any distribution function. Then the functional equation of type (1.13), reads

$$\bar{V}_p = \iint_{B_2} w\,.\,\Phi(V_p^0, V_M, \bar{V}_p, \bar{\alpha}, p)\,\mathrm{d}V_p^0\,\mathrm{d}V_M + \iint_{B_1} wV_p^0\,\mathrm{d}V_p^0\,\mathrm{d}V_M. \tag{6.12}$$

If our attention is not especially drawn to the initial deformation, i.e. if p is sufficiently large, the region of rigidity B_1 may be disregarded at all. Namely, an estimate gives $\alpha^{**} < 0{\cdot}01$ so far as $\bar{\mathbf{V}}_p/\bar{\mathbf{V}}_p^0 < 1-5k_1/\bar{\boldsymbol{\alpha}}$ which, in turn, is fulfilled in nearly all the cases, because of $k_1 \ll 1$. Therefore we are faced with the problem of solving the implicite equation for $\bar{\mathbf{V}}_p$

$$\bar{\mathbf{V}}_p = \iint w(V_p^0, V_M)\Phi(V_p^0, V_M, \bar{\mathbf{V}}_p, \bar{\boldsymbol{\alpha}}, p)\,\mathrm{d}V_p^0\,\mathrm{d}V_M \equiv \psi(\bar{\mathbf{V}}_p). \tag{6.13}$$

Suppose for the moment that the function $\Phi(V_p^0, V_M, \bar{\mathbf{V}}_p, \bar{\boldsymbol{\alpha}}, p)$ is known. Then the problem may be attacked by means of successive iteration by applying the general scheme of solving an equation

$$x = h(x)$$

by means of successive iteration:

$$x_{n+1} = h(x_n).$$

As is well known the condition of convergence is

$$\left|\frac{\mathrm{d}h}{\mathrm{d}x}\right| < 1$$

which, in the case of (6.13), takes the form

$$1 > \left|\frac{\mathrm{d}\psi}{\mathrm{d}\bar{\mathbf{V}}_p}\right| = \left|\iint w(V_p^0, V_M)\frac{\mathrm{d}\Phi}{\mathrm{d}\bar{\mathbf{V}}_p}\,\mathrm{d}V_p^0\,\mathrm{d}V_M\right| = \left|\overline{\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\bar{\mathbf{V}}_p}\right)}\right|. \tag{6.14}$$

But that proves right. According to (6.9)

$$\bar{\mathbf{V}}_p = \bar{\mathbf{V}}_p^0 - \frac{\bar{\mathbf{V}}_p^0+\bar{\mathbf{V}}_M}{V_p^0+V_M}(V_p^0-V_p) + k_1(\bar{\mathbf{V}}_p^0+\bar{\mathbf{V}}_M)\left(z_1 - \ln\frac{V_p+V_M}{V_p}\right)$$

holds, and therefore

$$\left|\frac{\mathrm{d}\Phi}{\mathrm{d}\bar{\mathbf{V}}_p}\right| = \left|\left(\frac{\mathrm{d}\bar{\mathbf{V}}_p}{\mathrm{d}V_p}\right)^{-1}\right| = \left|\left(\frac{\bar{\mathbf{V}}_p^0+\bar{\mathbf{V}}_M}{V_p^0+V_M} + k_1(\bar{\mathbf{V}}_p^0+\bar{\mathbf{V}}_M)\frac{V_M}{(V_p+V_M)V_p}\right)^{-1}\right| < \frac{V_p^0+V_M}{\bar{\mathbf{V}}_p^0+\bar{\mathbf{V}}_M}$$

$$\left|\overline{\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\bar{\mathbf{V}}_p}\right)}\right| < 1.$$

Having (6.9) rewritten as

$$V_p = V_p^0 - \frac{V_p^0+V_M}{\bar{\mathbf{V}}_p^0+\bar{\mathbf{V}}_M}(\bar{\mathbf{V}}_p^0-\bar{\mathbf{V}}_p) + k_1(V_p^0+V_M)\left(\ln\frac{V_p+V_M}{V_p} - z_1\right) \equiv \Phi^*(V_p) \tag{6.15}$$

the same approximation method may be applied for calculating (6.11). Convergence requires here $V_p > k_1(V_M+V_p^0)$. Otherwise, one has to start from $V_p = \Phi^{-1*}(V_p)$.

A numerical evaluation of the equations (6.9) and (6.13) (with the help of the method described above) can be found in Section 7.

In some cases the numerical solution of (6.9) and (6.13) may be avoided and replaced by another approximation. Observing (1.10 and 1.8) the first and second term of (6.15) reads

$$V_p^0 - \frac{V_p^0+V_M}{\bar{\mathbf{V}}_p^0+\bar{\mathbf{V}}_M}(\bar{\mathbf{V}}_p^0-\bar{\mathbf{V}}_p) = A\left(y, \frac{V_p^0}{V_M}\right)\cdot V_M$$

where

$$A = \frac{V_p^0}{V_M}(1-\bar{\alpha}(1-y))-\bar{\alpha}(1-y). \tag{6.16}$$

Now according to (6.15)

$$V_p = A\left(y, \frac{V_p^0}{V_M}\right)V_M \tag{6.17}$$

turns out to be a good zeroth approximation for (6.11) (because of $k_1 \ll 1$) so far as $A(y, V_p^0/V_M) \gtrsim 0{\cdot}1$ i.e.

$$\frac{V_p^0}{V_M} \gtrsim \frac{0{\cdot}1+\bar{\alpha}(1-y)}{1+\bar{\alpha}(1-y)}.$$

If, for instance $\bar{\alpha} = 0{\cdot}5$, $y = 0{\cdot}5$ the requirement is $V_p^0/V_M \gtrsim 0{\cdot}2$. In other words, the approximation (6.17) may be accepted for pores whose initial volume is sufficiently large. Further, taken for granted that very small pores can be disregarded at all, we are able to proceed to a good approximation solution of our actual problem.

For this end, let us perform the average of (6.15) to obtain

$$\overline{(V_p^0+V_M)\left(\ln\frac{V_p+V_M}{V_p}-z_1\right)} = 0$$

in other terms

$$z_1 = \frac{3p}{2\sigma_0} = \frac{1}{\overline{V}_p^0+\overline{V}_M}\overline{(V_p^0+V_M)\ln\left(1+\frac{V_M}{V_p}\right)} \tag{6.18}$$

However, so far as the quoted suppositions are valid, it is allowed to substitute (6.17), which furnishes the approximative solution

$$z_1 = \frac{3p}{2\sigma_0} = \frac{1}{\overline{V}_p^0+\overline{V}_M}\overline{(V_p^0+V_M)\ln\left(1+\frac{1}{A(y, V_p^0/V_M)}\right)} \tag{6.19}$$

Finally, consider the special case that all the pores have the same value of V_p^0/V_M Then A is constant under averaging, namely, $\overline{V}_p/\overline{V}_M$, (see 6.17). Hence, (6.19) goes over to Torre's formula (6.6), in accordance with the discussion in the context of (1.1). In terms of physics, such a constancy of the pores means, that their characteristics V_p, V_p^0, V_M do not fluctuate too much around their mean value. Therefore, Torre's formula is obtained directly from (6.18) if products of those fluctuations were neglected.

The discussion of unloading is much more complicated because of the impossibility to evaluate equation (6.12) exactly. Therefore, we have taken into consideration only Torre's formula (6.6). But in this case (disregarding elastic deformations) it is easily seen that redeformation of the pores does not occur.

7. NUMERICAL RESULTS

For the purpose of connecting the theoretical considerations with some experimental results, we have taken the Jüttner's [7] measurements, which concern pressing of metal powder. Here the compression is mainly a plastic process. On the other hand equations (6.13, 6.15 and 3.16), respectively, have been evaluated. Moreover, the formula (1.1), given by Merz and Pompe [2] has been treated numerically. So far as a distribution function for the pore size is needed we decided for Bockstiegel's [6] experimentally determined distribution. Bockstiegel has found that the accumulated porosity is of the log-normal type. This result corresponds to the following distribution function

$$w(V_p^0) = \frac{C}{V_{p\max}^0}\left(\frac{V_{p\max}^0}{V_p^0}\right)^2 \ln\left(\frac{V_{p\max}^0}{V_p^0}\right) \exp\left[-\left(\frac{\ln(V_{p\max}^0/V_p^0)}{\Delta}\right)^2\right] \tag{7.1}$$

where $C = 1{\cdot}02$ follows from

$$\int w(V_p^0)\,\mathrm{d}V_p^0 = 1 \tag{7.2}$$

$V_{p\max}^0$ is the maximal pore size ($w(V_{p\max}^0) = 0$).

Therefore the range of integration reduces to $0 \le V_p^0 \le V_{p\max}^0$, $\Delta = 0{\cdot}9$ has been taken from Bockstiegel's measurements. $V_{p\max}^0$ has been calculated from the condition that $\bar{V}_p^0/(\bar{V}_p^0+\bar{V}_M)$ equals the experimental $\bar{\alpha}$. Distribution functions for V_M were not known to us. For this reason we have taken $V_M = \bar{V}_M$ and with it the final distribution function reads:

$$w(V_p^0, V_M) = w(V_p^0)\delta(V_M - \bar{V}_M). \tag{7.3}$$

Further, the quantities σ_0 and σ_0^e, respectively, as well as $k = \sigma_0^e/(2G^e)$ and $k_1 = \sigma_0/(2G^e)$ have been fitted to the experimental data. Variation of k does not essentially affect the turn of the curves, in the case of model I (see 3.17). The iteration procedure for the solution of (6.15) in the case of $|\mathrm{d}\Phi^*/\mathrm{d}V_p^0| \approx 1$, $V_p \approx k_1(V_M + V_p^0)$ has been replaced by the Newtonian formula (for better convergence). Figure 2 shows the plots, achieved by the mentioned procedure, together with the experimental results.

8. CONCLUSIONS

Two models for the elastic–plastic compression of porous bodies have been examined. The first one is easier from the standpoint of analytical treatment. On the other hand, it is seen from Fig. 2 that the curves belonging to model II can be fitted to the experimental data in a better way than those which yield from the other model. Hence, the measurements seem to be in favour of model II. However, a final statement about the advantage of this model in favour of the other one cannot yet be given. This is so because any strain hardening has not been taken into consideration. Acceptance, for instance, of a linear hardening law would lead to a greater curvature of the curves calculated on the basis of model I, such that also in this case a better fitting to the experiments could be achieved.

Nevertheless, the results prove the method of self-consistency to be applicable with satisfactory success to the determination of elastic–plastic constitutive relations of porous bodies. Therefore the authors feel encouraged to an attempt of extending this method to more complicated stress-histories.

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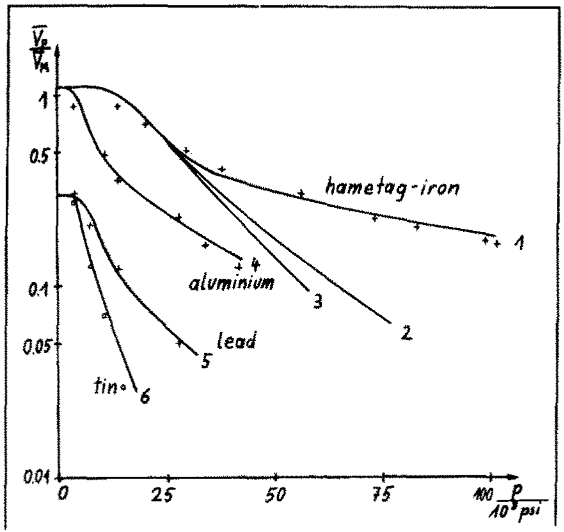

FIG. 2. Comparison with experiments.

1 model II	$\sigma_0 = 37 . 10^3$ psi	$k_1 = 0{\cdot}01$
2 eq. (1.1)	$\sigma_0 = 43 . 10^3$ psi	
3 model I	$\sigma_0^e = 13 . 10^3$ psi	$k = 0{\cdot}001$
4 model II	$\sigma_0 = 13 . 10^3$ psi	$k_1 = 0{\cdot}01$
5 model II	$\sigma_0 = 14 . 10^3$ psi	$k_1 = 0{\cdot}03$
6 model II	$\sigma_0 = 9 . 10^3$ psi	$k_1 = 0{\cdot}05$

REFERENCES

[1] C. TORRE, Theorie und Verhalten zusammengepreßteer Pulver. *Berg- u. Hüttenmännische Monatshefte* **93**, 62 (1948).
[2] A. MERZ and W. POMPE, Isostatisches Pressen von Metallpulvern. *Freiberger Forschungshefte* **A 501**, 63 (1971).
[3] T. Y. CHU and Z. HASHIN, Plastic behavior of composites and porous media under isotropic stresses. *Int. J. Engng. Sci.* **9**, 971 (1971).
[4] I. M. RYSHIK and I. S. GRADSTEIN, *Summen-, Produkte- und Integraltafeln*. Dt. Verlag d. Wiss (1955).
[5] W. PRAGER and P. G. HODGE, *Theorie ideal-plastischer Körper*. p. 110. Springer (1954).
[6] G. BOCKSTIEGEL, The porosity–pressure curve and its relation to the pore size distribution in iron powder compacts. *Int. Powder Metallurgy Conf.* New York (1965).
[7] C. JÜTTNER, Messungen zum Verdichtungsverhalten von Metallpulvern unter hydrostatischem Druck. Unpublished (1969).
[8] W. HERRMANN, Constitutive Equation for the Dynamic Compaction of Ductile Porous Materials. *J. appl. Phys.* **40**, 2490 (1969).
[9] M. CARROLL and A. C. HOLT, Suggested modification of the *P*–α model for porous materials. *J. appl. Phys.* **43**, 759 (1972).

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Абстракт—В предлагаемой работе даются зависмости между гидростатическим давлением и среднем объемом поры некоторого пористого тела. Главным образом, обращается внимание к пластической зоне истории деформации. В противоположности к предыдущим работам, принимается во внимание взаймодействие между разньми порами суммарным способом, путем применения само-плотного метода. С этой целью заменяется окружность отдельной поры однородным материалом, который описан законом сжатия. Определяются результаты численно и даются приближенные формулы. Сравнение с экспериментальными результатами указывает частично хорошое совпадение.